J. Fluid Mech. (2008), vol. 601, pp. 365–379.  
doi:10.1017/S0022112008000608 Printed in the United Kingdom

# A physically motivated approach for filtering acoustic waves from the equations governing compressible stratified flow

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(Received 29 December 2006 and in revised form 24 January 2008)

An incompressibility approximation is formulated for isentropic motions in a compressible stratified fluid by defining a pseudo-density $\rho^*$ and enforcing mass conservation with respect to $\rho^*$ instead of the true density. Using this approach, sound waves will be eliminated from the governing equations provided $\rho^*$ is an explicit function of the space and time coordinates and of entropy. By construction, isentropic pressure perturbations have no influence on the pseudo-density.

A simple expression for $\rho^*$ is available for perfect gases that allows the approximate mass conservation relation to be combined with the unapproximated momentum and thermodynamic equations to yield a closed system with attractive energy conservation properties. The influence of pressure on the pseudo-density, along with the explicit $(\boldsymbol{x}, t)$ dependence of $\rho^*$ is determined entirely by the hydrostatically balanced reference state.

Scale analysis shows that the pseudo-incompressible approximation is applicable to motions for which $\mathscr{M}^2 \ll \min(1, \mathscr{R}^2)$, where $\mathscr{M}$ is the Mach number and $\mathscr{R}$ the Rossby number. This assumption is easy to satisfy for small-scale atmospheric motions in which the Earth's rotation may be neglected and is also satisfied for quasi-geostrophic synoptic-scale motions, but not planetary-scale waves. This scaling assumption can, however, be relaxed to allow the accurate representation of planetary-scale motions if the pressure in the time-evolving reference state is computed with sufficient accuracy that the large-scale components of the pseudo-incompressible pressure represent small corrections to the total pressure, in which case the full solution to both the pseudo-incompressible and reference-state equations has the potential to accurately model all non-acoustic atmospheric motions.

## 1. Introduction

One of the simplest ways to filter sound waves from the equations governing fluid motion is through the Boussinesq approximation, which is widely used to facilitate both theoretical analysis and numerical computation. Greater errors are typically incurred when applying the Boussinesq approximation to gases than to single-constituent liquids, because liquids are nearly incompressible, whereas gases are not. Most of the additional error comes in the treatment of the mass continuity equation for gases, and that will be the main focus of this paper. Some of the earliest investigations of the conditions under which the Boussinesq approximation applies to compressible fluids were focused on atmospheric applications (Batchelor 1953), and such applications also provide the context for this paper. Nevertheless, our approach appears to be directly applicable to low-Mach-number† flow in a wide variety of stratified compressible fluids.

If the fluid is incompressible (i.e. if density is independent of pressure) and if the flow is adiabatic, the only way to change the density of a fluid parcel is through the diffusion of heat and trace constituents through the sides of the parcel, and in many applications the influence of such diffusion on density can be neglected (Batchelor 1967, p. 75). Thus, in adiabatic incompressible flow,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0. \tag{1.1}$$

Here $\mathrm{D}/\mathrm{D}t$ denotes the convective derivative,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla,$$

and $\boldsymbol{u} = (u, v, w)$ is the velocity vector with respect to the Cartesian coordinates $(x, y, z)$.

Using (1.1), the mass continuity equation

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \nabla\cdot\boldsymbol{u} = 0. \tag{1.2}$$

reduces to the simple requirement of non-divergent flow,

$$\nabla\cdot\boldsymbol{u} = 0. \tag{1.3}$$

Replacing the prognostic continuity equation (1.2) with the diagnostic condition (1.3) eliminates sound-wave (or elastic-wave) solutions from the governing equations and thereby allows the approximate governing equations to be integrated numerically using explicit time-differencing with a much larger time step than that required to preserve stability during integrations of the unapproximated system.

When modelling disturbances in gases, computational efficiency can be similarly enhanced by using a set of governing equations that does not support sound waves in those cases where sound waves make a negligible contribution to the total disturbance. Simply requiring the flow to be non-divergent is, however, a relatively crude way to eliminate sound waves from stratified compressible fluids. In particular, (1.3) cannot provide a good approximation to (1.2) unless the depth of the fluid is much shallower than the scale height of the undisturbed density field (Spiegel & Veronis 1960), but this is not the case for many types of atmospheric motions.

Considerable effort has therefore been devoted to deriving alternatives to (1.3) that eliminate sound waves while more accurately representing the behaviour of other low-Mach-number dynamical modes supported by the exact compressible equations. Since the vertical gradient of the reference-state density can be large, many such 'anelastic' systems approximate the mass continuity equation as

$$\frac{w}{\overline{\rho}}\frac{\mathrm{d}\overline{\rho}}{\mathrm{d}z} + \nabla\cdot\boldsymbol{u} = 0, \tag{1.4}$$

where $\overline{\rho}(z)$ is the vertically varying density in the reference state (Ogura & Phillips 1962; Lipps & Hemler 1982). When using this approach, some care is required in the approximation of the pressure gradient terms in the momentum equations to ensure that the resulting system is energy conservative (Lipps 1990).

† The Mach number $\mathscr{M}$ is the characteristic velocity scale divided by the speed of sound.

An alternative approach suggested by Durran (1989) is to define a pseudo-density $\rho^*$ and to enforce mass conservation with respect to this pseudo-density such that

$$\frac{1}{\rho^*}\frac{\mathrm{D}\rho^*}{\mathrm{D}t} + \nabla\cdot\boldsymbol{u} = 0. \tag{1.5}$$

In the case of perfect gases, the pseudo-density may be defined as $\rho^* = \overline{\rho}\overline{\theta}/\theta$, where $\theta = T\pi^{-1}$ is the potential temperature, $T$ is the temperature, $\pi = (p/p_s)^{R/c_p}$, $c_p$ is the specific heat at constant pressure, $R$ is the gas constant, $p_s$ is a constant reference pressure, and $\overline{\theta}(z)$ is the potential temperature of the reference state. Using this definition for $\rho^*$, (1.5) may be combined with the unapproximated momentum and thermodynamic equations to yield an energy-conservative system that does not support sound waves. Substituting $\rho^* = \overline{\rho}\overline{\theta}/\theta$ into (1.5) and using the thermodynamic equation

$$\frac{\mathrm{D}S}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}(c_p \ln\theta) = \frac{H_m}{T}, \tag{1.6}$$

where $S$ is entropy and $H_m$ is the heating rate per unit mass, (1.5) may be alternatively expressed as the diagnostic relation

$$\nabla\cdot(\overline{\rho}\overline{\theta}\boldsymbol{u}) = \frac{\rho^* H_m}{c_p\pi} \approx \frac{\overline{\rho}H_m}{c_p\overline{\pi}}. \tag{1.7}$$

Durran (1989) referred to the replacement of the true continuity equation by (1.5) as the 'pseudo-incompressible approximation' and showed that it is valid when the Mach number is small and $\pi' \ll \overline{\pi}$.

In the following we will examine how the concept of pseudo-density can be generalized to allow (1.5) to more closely approximate the true mass conservation equation while continuing to filter sound waves. The most general functional form of $\rho^*$ sufficient to filter sound waves is considered in § 2. A scale analysis revealing the conditions under which $\rho^*$ may include explicit dependence on all space and time coordinates is presented in § 3. The energy conservation properties of the generalized pseudo-incompressible system are examined in § 4. Auxiliary relations such as potential vorticity conservation in discussed in § 5. The relation between the pseudo-incompressible approximation and other anelastic systems is considered in § 6. Section 7 contains the conclusions.

## 2. Elimination of sound waves

Consider sound-wave propagation in a stably stratified single-constituent fluid (i.e. one whose equilibrium thermodynamic state is uniquely determined by the value of two state variables). For such a fluid the density may be expressed as $\rho(p, S)$, and

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \left(\frac{\partial\rho}{\partial p}\right)_S \frac{\mathrm{D}p}{\mathrm{D}t} + \left(\frac{\partial\rho}{\partial S}\right)_p \frac{\mathrm{D}S}{\mathrm{D}t}.$$

Using the preceding equation, (1.2), and the definition of the speed of sound $c^2 = (\partial p/\partial\rho)_S$, the mass continuity relation for isentropic motions reduces to

$$\frac{1}{\rho c^2}\frac{\mathrm{D}p}{\mathrm{D}t} + \nabla\cdot\boldsymbol{u} = 0. \tag{2.1}$$

Taking $c$ as constant and linearizing (2.1) about a resting hydrostatically balanced basic state $\overline{\rho}(z)$ and $\overline{p}(z)$, gives

$$\frac{1}{c^2}\frac{\partial p'}{\partial t} + \nabla\cdot(\overline{\rho}\boldsymbol{u}') = -\frac{\overline{\rho}N^2 w'}{g}, \tag{2.2}$$

where

$$N^2 = -g\left(\frac{1}{\overline{\rho}}\frac{\mathrm{d}\overline{\rho}}{\mathrm{d}z} + \frac{g}{c^2}\right)$$

is the Brunt–Väisälä frequency. A similar linearization of the momentum equation for inviscid flow,

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p = -\rho g\boldsymbol{k}, \tag{2.3}$$

gives

$$\frac{\partial\overline{\rho}\boldsymbol{u}'}{\partial t} + \nabla p' = -g\rho'\boldsymbol{k}. \tag{2.4}$$

Eliminating $\overline{\rho}\boldsymbol{u}'$ between (2.2) and (2.4), one obtains

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = g\frac{\partial\rho'}{\partial z} - \frac{\overline{\rho}N^2}{g}\frac{\partial w'}{\partial t}. \tag{2.5}$$

Sound waves are solutions to the homogeneous part of (2:5). Pressure changes due to buoyancy (i.e. gravity) waves are forced through the right-hand side of (2.5).

Now suppose that true mass conservation is approximated using the pseudo-incompressible continuity equation (1.5). As an alternative to the true density, which can be expressed as a function of any two distinct thermodynamic state variables, suppose the pseudo-density has the functional form $\rho^*(Q, x, y, z, t)$, where $Q$ is some thermodynamic state variable. Noting that $Q$ may be expressed as a function of just pressure and entropy since the medium in question is a single-constituent fluid, the convective derivative of $\rho^*$ becomes

$$\frac{\mathrm{D}\rho^*}{\mathrm{D}t} = \frac{\partial\rho^*}{\partial Q}\left[\left(\frac{\partial Q}{\partial p}\right)_S\frac{\mathrm{D}p}{\mathrm{D}t} + \left(\frac{\partial Q}{\partial S}\right)_p\frac{\mathrm{D}S}{\mathrm{D}t}\right] + \left(\frac{\partial\rho^*}{\partial t}\right)_Q + \boldsymbol{u}\cdot\nabla_Q\rho^*, \tag{2.6}$$

where $\nabla_Q$ indicates that the gradient is taken at constant $Q$. For isentropic flow and this choice for $\rho^*$, the pseudo-incompressible continuity equation (1.5) becomes

$$\frac{\partial\rho^*}{\partial Q}\left(\frac{\partial Q}{\partial p}\right)_S\frac{\mathrm{D}p}{\mathrm{D}t} + \left(\frac{\partial\rho^*}{\partial t}\right)_Q + \boldsymbol{u}\cdot\nabla_Q\rho^* + \rho^*\nabla\cdot\boldsymbol{u} = 0. \tag{2.7}$$

As evident from the derivation of (2.5), sound waves will be eliminated if the first term in (2.7) is zero, or equivalently, if $Q$ is solely a function of $S$. Thus, $\rho^*(S, x, y, z, t)$ is the most general functional form for $\rho^*$ that will filter sound waves via the approximate mass conservation relation (1.5).

Although there are many expressions for $\rho^*$ that filter sound waves, only those that allow the accurate approximation of low-Mach-number motions in a compressible fluid are of practical interest. In anelastic systems the speed of sound is effectively infinite as adjustments to convergent or divergent forcing occur instantaneously throughout the fluid. Eliminating sound waves by taking the $c\rightarrow\infty$ limit in (2.2) yields an expression for the velocity divergence that may be further approximated to obtain the Boussinesq expression (1.3) or the Ogura & Phillips (1962) continuity equation (1.4). One can avoid such additional approximations by choosing an expression for $\rho^*$

for use in the pseudo-incompressible continuity equation that allows (1.5) to simplify to an expression retaining all the terms in (2.2) that can be retained while still eliminating sound waves.

As a preliminary step toward the construction of such $\rho^*$, it is helpful to express $N^2$ in terms of the vertical entropy gradient in a resting hydrostatically balanced reference state. Writing $\rho$ as a function of $p$ and $S$, the change in reference-state density with height is

$$\frac{\mathrm{d}\overline{\rho}}{\mathrm{d}z} = \left(\frac{\partial \overline{\rho}}{\partial \overline{p}}\right)_S \frac{\mathrm{d}\overline{p}}{\mathrm{d}z} + \left(\frac{\partial \overline{\rho}}{\partial \overline{S}}\right)_p \frac{\mathrm{d}\overline{S}}{\mathrm{d}z} = \frac{1}{c^2}\frac{\mathrm{d}\overline{p}}{\mathrm{d}z} + \left(\frac{\partial \overline{\rho}}{\partial \overline{S}}\right)_p \frac{\mathrm{d}\overline{S}}{\mathrm{d}z},$$

and the density change in a parcel displaced isentropically is

$$\mathrm{d}\rho = \left(\frac{\partial \rho}{\partial p}\right)_S \mathrm{d}p = \frac{1}{c^2}\,\mathrm{d}p.$$

If the parcel is displaced vertically by $\mathrm{d}z$ and its pressure adjusts to match the pressure in its environment (so that $\mathrm{d}p = \mathrm{d}\overline{p}$), the density difference between the parcel and its environment is

$$\mathrm{d}\rho - \mathrm{d}\overline{\rho} = \frac{1}{c^2}\left(\mathrm{d}p - \frac{\mathrm{d}\overline{p}}{\mathrm{d}z}\,\mathrm{d}z\right) - \left(\frac{\partial \overline{\rho}}{\partial \overline{S}}\right)_p \frac{\mathrm{d}\overline{S}}{\mathrm{d}z}\,\mathrm{d}z = -\left(\frac{\partial \overline{\rho}}{\partial \overline{S}}\right)_p \frac{\mathrm{d}\overline{S}}{\mathrm{d}z}\,\mathrm{d}z.$$

The Brunt–Väisälä frequency squared is the negative of the buoyancy restoring force per unit vertical displacement (Gill 1982, p. 51), so

$$N^2 = \frac{g}{\overline{\rho}}\frac{\mathrm{d}}{\mathrm{d}z}(\rho - \overline{\rho}) = g\overline{\alpha}\frac{\mathrm{d}\overline{S}}{\mathrm{d}z}, \quad \text{where} \quad \alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial S}\right)_p.$$

Now suppose $\rho^* = \overline{\rho}\mathrm{e}^{\beta(\overline{S}-S)}$, where $\beta$ is a constant and the overbars denote fields associated with the same vertically varying hydrostatically balanced basic state used to linearize the compressible equations. Then evaluating (2.7) with $Q = S$ yields

$$\nabla\cdot(\overline{\rho}\mathrm{e}^{\beta\overline{S}}\boldsymbol{u}) = 0, \tag{2.8}$$

which linearizes to

$$\beta\overline{\rho}\frac{\mathrm{d}\overline{S}}{\mathrm{d}z}w' + \nabla\cdot(\overline{\rho}\boldsymbol{u}') = 0. \tag{2.9}$$

If the fluid is a perfect gas, $S = c_p \ln\theta$ and $\alpha = c_p^{-1}$ is a constant, so one may choose $\beta = \alpha$ and reduce (2.9) to

$$\nabla\cdot(\overline{\rho}\boldsymbol{u}') = -\frac{\overline{\rho}N^2 w'}{g}, \tag{2.10}$$

which is the desired $c \to \infty$ limit of the full compressible relation (2.2). If the fluid is not a perfect gas, (2.10) might still be approximately satisfied by setting $\beta$ equal to a constant representative value for $\alpha$. Generalizations to other equations of state have been discussed by Almgren *et al.* (2006), who extended the Durran (1989) pseudo-incompressible system to describe stellar atmospheres.

## 3. A generalized pseudo-incompressible system for perfect gases

The functional form of the pseudo-density suggested for perfect gases by Durran (1989) was $\rho^*(\theta, z)$. Almgren (2000) noted that the scale analysis used to derive the pseudo-incompressible system can be generalized to include explicit time dependence in $\rho^*$ and the hydrostatically balanced reference state. A further generalization to include horizontal variations is also possible, and will be included here. Allowing horizontal variations in a steady reference state can be useful, for example, in global atmospheric modelling because there are significant equator-to-pole variations in the zonal-mean thermodynamic fields. More generally, the reference state could be the time-evolving flow from a low-resolution global spectral model that solves the compressible hydrostatic governing equations, in which case a complete description of the atmospheric state would be provided by the sum of the reference-state and pseudo-incompressible pressure fields together with the remaining variables from the pseudo-incompressible solution.

Let $\tilde{\pi}$, $\tilde{\theta}$ and $\tilde{\rho}$ define a spatially varying reference state in hydrostatic balance

$$c_p\tilde{\theta}\frac{\partial\tilde{\pi}}{\partial z} = -g, \tag{3.1}$$

and suppose that

$$\rho^* = \frac{\tilde{\rho}(x, y, z, t)\tilde{\theta}(x, y, z, t)}{\theta}. \tag{3.2}$$

Then substituting (2.6) into (1.5) with $Q = \theta$ yields

$$\frac{1}{\rho^*}\left[\frac{\partial\rho^*}{\partial\theta}\frac{\mathrm{D}\theta}{\mathrm{D}t} + \frac{1}{\theta}\frac{\partial\tilde{\rho}\tilde{\theta}}{\partial t} + \frac{\boldsymbol{u}}{\theta}\boldsymbol{\cdot}\nabla(\tilde{\rho}\tilde{\theta})\right] + \nabla\boldsymbol{\cdot}\boldsymbol{u} = 0.$$

Using the thermodynamic equation (1.6) and approximating the full temperature as that of the reference state ($T \approx \tilde{T} = \tilde{\pi}\tilde{\theta}$), the preceding equation simplifies to

$$\frac{\partial\tilde{\rho}\tilde{\theta}}{\partial t} + \nabla\boldsymbol{\cdot}(\tilde{\rho}\tilde{\theta}\boldsymbol{u}) = \frac{\tilde{\rho}H_m}{c_p\tilde{\pi}}. \tag{3.3}$$

Define perturbations with respect to the reference state such that $\theta' = \theta - \tilde{\theta}$ and $\pi' = \pi - \tilde{\pi}$, and for computational accuracy and notational convenience, separate $\tilde{\pi}$ into a large horizontally uniform component $\tilde{\pi}_v(z, t)$ plus a remainder $\tilde{\pi}_h(x, y, z, t)$. The pseudo-incompressible system associated with the choice of (3.2) for $\rho^*$ may be expressed as (3.3) together with

$$\frac{\mathrm{D}\boldsymbol{u}_h}{\mathrm{D}t} + f\boldsymbol{k}\times\boldsymbol{u}_h + c_p\theta\nabla_h(\tilde{\pi}_h + \pi') = 0, \tag{3.4}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} + c_p\theta\frac{\partial\pi'}{\partial z} = g\frac{\theta'}{\tilde{\theta}}, \tag{3.5}$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{H_m}{c_p\tilde{\pi}}, \tag{3.6}$$

where $\boldsymbol{u}_h$ is the horizontal velocity vector, $\nabla_h$ is the horizontal gradient operator, and $f$ is the Coriolis parameter. The exact expressions for the Coriolis acceleration could alternatively have been included in (3.4)–(3.5), but the notation is simplified by retaining only those terms arising from the component of the Earth's angular velocity parallel to the local vertical coordinate and these are the only terms that are important for the following scale analysis.

Except for the representation of the Coriolis force, the horizontal momentum equation (3.4) is exact. The vertical momentum equation (3.5) is similarly exact; gravity and the vertical pressure gradient of the reference state have simply been cancelled out (and use made of $c_p\theta'\partial\tilde{\pi}/\partial z = -g\theta'/\tilde{\theta}$). The thermodynamic equation contains an approximation in the diabatic heating term, where $\pi$ has been replaced by $\tilde{\pi}$ that would be justified if $\pi' \ll \tilde{\pi}$.

What conditions, other than the smallness of $\pi'$, are required to justify the simplification of the full compressible equations to the system consisting of (3.3)–(3.6)? Since all the remaining approximations are contained in the pseudo-incompressible continuity equation ((1.5) or equivalently (3.3)) we focus on the conditions under which (3.3) provides a good approximation to the corresponding relation in the fully compressible case. The exact equation of state may be written

$$\pi = \left(\frac{R}{p_s}\rho\theta\right)^{R/c_v}, \tag{3.7}$$

implying that

$$\frac{c_v}{R\pi}\frac{\mathrm{D}\pi}{\mathrm{D}t} = \frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \frac{1}{\theta}\frac{\mathrm{D}\theta}{\mathrm{D}t}.$$

Using the exact mass continuity and thermodynamic equations, the preceding equation is

$$\frac{c_v}{R\pi}\frac{\mathrm{D}\pi}{\mathrm{D}t} + \nabla\cdot\boldsymbol{u} = \frac{H_m}{c_p\theta\pi},$$

or alternatively

$$\frac{c_v}{R\pi}\frac{\mathrm{D}\pi'}{\mathrm{D}t} + \frac{c_v}{R\pi}\frac{\mathrm{D}\tilde{\pi}}{\mathrm{D}t} + \nabla\cdot\boldsymbol{u} = \frac{RH_m}{c_pT}. \tag{3.8}$$

Neglecting the first term, approximating $\pi$ by $\tilde{\pi}$ in the second term, and approximating the temperature in the heating term by $\tilde{T}$, (3.8) takes the form

$$\frac{1}{\tilde{\rho}\tilde{\theta}}\frac{\partial\tilde{\rho}\tilde{\theta}}{\partial t} + \frac{\boldsymbol{u}}{\tilde{\rho}\tilde{\theta}}\cdot\nabla(\tilde{\rho}\tilde{\theta}) + \nabla\cdot\boldsymbol{u} = \frac{H_m}{c_p\tilde{\theta}\tilde{\pi}}, \tag{3.9}$$

which is equivalent to (3.3). Thus the only approximations incorporated in the adiabatic pseudo-incompressible system are that $\pi' \ll \tilde{\pi}$ and that the the first term in (3.8) is small. In the diabatic case, the approximation $T \approx \tilde{T}$ is also used to evaluate the heating term, but this approximation is not likely to be an important source of error because in many practical applications the heating term is not known exactly, but rather given by an approximation to the relevant physical processes (such as latent heat release in clouds or radiative flux divergences). In addition, the following scale analysis implies that $O(T'/\tilde{T}) = O(\pi'/\tilde{\pi})$.

When is the first term in (3.8) small? Define the following length scales and non-dimensional variables (indicated by $\hat{}$) to assess the magnitudes of the terms in (3.4), (3.5) and (3.8): $z = D\hat{z}$, $(u, v) = U(\hat{u}, \hat{v})$, $w = W\hat{w}$, $c_p\theta = c^2\hat{\theta}$. Also let $(x, y) = L(\hat{x}, \hat{y})$, where $L$ is a length scale for the horizontal variation of all fields except $\tilde{\pi}_h$, which is assumed to have horizontal scale $\tilde{L}$. Take the scales related to $\pi$ as $\tilde{\pi}_v = 1 \times \hat{\tilde{\pi}}_v$, $\tilde{\pi}_h = \tilde{\Pi}_h\hat{\tilde{\pi}}_h$, and $\pi' = \Pi\hat{\pi}$; typically it is possible to decompose $\tilde{\pi}$ such that $\tilde{\Pi}_h \ll 1$. Finally, define the aspect ratio $\delta = D/L$, the Mach number $\mathscr{M} = U/c$ and the Rossby number $\mathscr{R} = U/(f_0L)$, where $f_0$ is the scale for the Coriolis parameter.

Using the advective time scale $t = (L/U)\hat{t}$, defining $\sigma = L/\tilde{L}$, and assuming $W/D \leqslant U/L$, the horizontal momentum equation (3.4) may be expressed in non-dimensional form as

$$\frac{\mathrm{D}\hat{\boldsymbol{u}}_h}{\mathrm{D}\hat{t}} + \mathscr{R}^{-1}\boldsymbol{k}\times\hat{\boldsymbol{u}}_h + \mathscr{M}^{-2}[\sigma\tilde{\Pi}_h\hat{\theta}\hat{\tilde{\nabla}}_h\hat{\tilde{\pi}}_h + \Pi\hat{\theta}\hat{\nabla}_h\hat{\pi}] = 0. \tag{3.10}$$

If the reference-state horizontal pressure gradients are no larger than those arising from variations in the pseudo-incompressible pressure ($\tilde{\Pi}_h/\tilde{L} \leqslant \Pi/L$, or $\sigma\tilde{\Pi}_h \leqslant \Pi$), balances for cases with non-trivial flow are achieved in (3.10) when the perturbation pressure scales such that

$$\Pi = \frac{\mathscr{M}^2}{\min(1, \mathscr{R})}. \tag{3.11}$$

Defining a time scale for the reference-state fluctuations such that $t = (\tilde{L}/U)\hat{\tilde{t}}$, the non-dimensional form of (3.8) may be expressed as

$$\Pi \frac{\mathrm{D}\hat{\pi}}{\mathrm{D}\hat{t}} + \sigma\tilde{\Pi}_h \frac{\mathrm{D}\hat{\tilde{\pi}}_h}{\mathrm{D}\hat{\tilde{t}}} + \left(\frac{WL}{DU}\right)\left[\frac{\partial \hat{\tilde{\pi}}_v}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{z}}\right] + \hat{\nabla}_h \cdot \hat{\boldsymbol{u}}_h = 0. \tag{3.12}$$

Now consider the circumstances in which the first term in the preceding can be neglected. For mesoscale and tropical circulations $\mathscr{R} \geqslant O(1)$, $W/D = U/L$ and the first term in (3.12) will be small compared to the last two when $\Pi = \mathscr{M}^2 \ll 1$, i.e. when the Mach number squared is much less than unity, which is easily satisfied in all meteorologically significant circulations. In the quasi-geostrophic limit ($\mathscr{R} \ll 1$), both $(WL)/(DU)$ and the horizontal divergence are $O(\mathscr{R})$ (Pedlosky 1987, sec. 6.3) and the first term in (3.12) will be small compared to the last two when $\Pi = \mathscr{M}^2\mathscr{R}^{-1} \ll \mathscr{R}$, implying that $\mathscr{M}^2 \ll \mathscr{R}^2$.

Thus, in those cases where the reference-state horizontal pressure gradients are no larger than those for the pseudo-incompressible pressure, the approximation of (3.8) by (3.9) is justified if

$$\mathscr{M}^2 \ll \min(1, \mathscr{R}^2). \tag{3.13}$$

This criterion is satisfied by typical mid-latitude synoptic-scale motions for which $\mathscr{M}^2 = 10^{-3}$ and $\mathscr{R}^2 = 10^{-2}$; however the approximation required to obtain the pseudo-incompressible system becomes more difficult to satisfy as the scale of the circulations becomes larger. For example, if $\mathscr{R} \ll 1$, (3.13) may be alternatively expressed as $(Lf_0/c)^2 \ll 1$, which is not satisfied for planetary waves.

On the other hand, the scale analysis supporting the approximation of (3.8) by (3.9), i.e. the neglect of the first term in (3.12), becomes trivial if the reference-state horizontal pressure gradients are much larger than those for the pseudo-incompressible pressure (i.e. if $\sigma\tilde{\Pi}_h \gg \Pi$), and this can be the case for large-scale circulations if the reference-state pressure is determined by the time-dependent solution to the compressible hydrostatic governing equations. In this case the pseudo-incompressible pressure can simply be the small correction required to account for non-hydrostatic effects in an essentially hydrostatic circulation.

To examine when reference-state horizontal pressure gradients might be dominant, define a Froude number as $\mathscr{F} = \sqrt{gD}/U$ and let $\epsilon$ be the scale for $\theta'/\tilde{\theta}$; then the non-dimensional vertical momentum equation (3.5) may be expressed in the form

$$\min(1, \mathscr{R})\mathscr{M}^2\delta^2 \frac{\mathrm{D}\hat{w}}{\mathrm{D}\hat{t}} + \Pi \frac{\partial \hat{\pi}}{\partial \hat{z}} = \mathscr{F}^2\mathscr{M}^2\epsilon \frac{\hat{\theta}}{\hat{\tilde{\theta}}}, \tag{3.14}$$

where once again the we have used $(WL)/(DU) = \min(1, \mathscr{R})$. Thus, the hydrostatic part of the pseudo-incompressible pressure scales like $\mathscr{F}^2\mathscr{M}^2\epsilon$, and the scale for the total pseudo-incompressible pressure (hydrostatic plus dynamic) satisfies the inequality

$$\Pi \geqslant \min(1, \mathscr{R})\mathscr{M}^2\delta^2. \tag{3.15}$$

When $\sigma\tilde{\Pi}_h \gg \Pi$, a non-trivial balance in the horizontal momentum equation (3.10) requires that $\sigma\tilde{\Pi}_h$ replace $\Pi$ in (3.11), which in combination with (3.15) implies

$$\frac{\mathscr{M}^2}{\min(1,\mathscr{R})} = \sigma\tilde{\Pi}_h \gg \Pi \geqslant \min(1,\mathscr{R})\mathscr{M}^2\delta^2. \tag{3.16}$$

For $\mathscr{R} \ll 1$, the preceding relation requires $\mathscr{R}^2\delta^2 \ll 1$, which is easily satisfied by synoptic-scale atmospheric motions for which $\mathscr{R} = 0.1$ and $\delta = 10^{-2}$. For small-scale motions $\mathscr{R} \geqslant O(1)$ and $\delta = 1$, so (3.16) is not satisfied and, as expected, the reference-state hydrostatic pressure gradient cannot dominate the pseudo-incompressible pressure gradient.

In summary, when the Earth's rotation is not dominant ($\mathscr{R} \geqslant O(1)$) both $\pi'/\tilde{\pi}$ and the ratio of the first term in (3.8) to the terms involving the velocity divergence are $O(\mathscr{M}^2)$, so the pseudo-incompressible approximation is justified if the Mach number is small. If $\mathscr{R} \ll 1$, the pseudo-incompressible approximation can potentially be justified in two different ways. If the reference-state horizontal pressure gradients are no larger than those associated with the pseudo-incompressible pressure $\pi'$, then both $\pi'/\tilde{\pi} \ll 1$ and the first term in (3.8) can be neglected if $\mathscr{M}^2 \ll \mathscr{R}^2$. If, on the other hand, the reference-state horizontal pressure gradients dominate those associated with $\pi'$, then $\sigma\tilde{\Pi}_h \gg \Pi$ and the approximations required to obtain the pseudo-incompressible equations are trivially satisfied. One cannot, however, expect the reference state to dominate the horizontal pressure gradients associated with some particular circulation unless $\tilde{\pi}_h$ is being computed with a time-dependent hydrostatic model with sufficient resolution to resolve the circulation in question, and even under such circumstances, special numerical techniques may be required to ensure that the pseudo-incompressible pressure correction remains small. Nevertheless, the calculation of such a time-dependent reference-state pressure provides a possible way to extend the pseudo-incompressible system to problems for which $\mathscr{M} \geqslant \mathscr{R}$.

## 4. Energy conservation

Now consider the energy conservation properties of (3.3)–(3.6) for isentropic flow. In the following, the hydrostatic reference state is not removed from the momentum equations so that $\pi = \tilde{\pi} + \pi'$ is the full Exner function pressure. Let $M = \boldsymbol{u}\cdot\boldsymbol{u}/2 + gz$ be the mechanical energy per unit mass, then the dot product of $\boldsymbol{u}$ with the momentum equations yields

$$\frac{\partial M}{\partial t} + \boldsymbol{u}\cdot\nabla M + c_p\theta\boldsymbol{u}\cdot\nabla\pi = 0.$$

Multiplying the preceding equation by $\rho^*$ and adding $M$ times (1.5) gives

$$\frac{\partial \rho^* M}{\partial t} + \nabla\cdot(\rho^* M\boldsymbol{u}) + c_p\rho^*\theta(\boldsymbol{u}\cdot\nabla\pi) = 0, \tag{4.1}$$

which is the mechanical energy equation for the pseudo-incompressible system. It differs from the corresponding relation in the full compressible system only in that $\rho$ is replaced by $\rho^*$.

Replacing $\tilde{\rho}\tilde{\theta}$ by $\rho^*\theta$ in (3.3), multiplying the result by $c_p\pi$ and adding it to (4.1) one obtains

$$\frac{\partial \rho^* M}{\partial t} + c_p\pi\frac{\partial \rho^*\theta}{\partial t} + \nabla\cdot[\rho^*(M + c_p\pi\theta)\boldsymbol{u}] = 0. \tag{4.2}$$

Using (3.7) and the definition of $\pi$

$$c_p\pi\frac{\partial\rho^*\theta}{\partial t}=\frac{\pi}{\tilde{\pi}}c_p\tilde{\pi}\frac{\partial\tilde{\rho}\tilde{\theta}}{\partial t}=\frac{\pi}{\tilde{\pi}}\frac{\partial}{\partial t}(c_v\tilde{\rho}\tilde{T}),$$

and noting that the unapproximated equation of state implies $c_p\pi\theta=c_vT+p/\rho$, (4.2) may be expressed as

$$\frac{\pi'}{\tilde{\pi}}\frac{\partial}{\partial t}(\tilde{\rho}c_v\tilde{T})+\frac{\partial}{\partial t}(\rho^*M+\tilde{\rho}c_v\tilde{T})+\nabla\cdot\left[\rho^*\left(M+c_vT+\frac{p}{\rho}\right)\boldsymbol{u}\right]=0. \tag{4.3}$$

In comparison, the total energy equation for the full compressible system is

$$\frac{\partial E}{\partial t}+\nabla\cdot\boldsymbol{F}=0,$$

where

$$E=\rho(M+c_vT),\quad \boldsymbol{F}=\boldsymbol{u}(E+p).$$

The mechanical energy density $\rho^*M$ and the energy flux densities in the pseudo-incompressible energy equation (4.3) differ from those in the exact system only by the factor $\rho^*/\rho$. The internal energy density in the pseudo-incompressible system $\tilde{\rho}c_v\tilde{T}$ is just the internal energy of the hydrostatic reference state.

The first term in (4.3) is exactly zero if the reference state is steady. In this case the differences between the exact and pseudo-incompressible energy equations are almost intuitive: (i) $\rho^*$ replaces $\rho$ in the expressions for the energy and energy-flux densities, (ii) since the momentum and thermodynamic equations are unapproximated, the total energy flux per unit mass is exact, but (iii) since sound waves have been filtered by the pseudo-incompressible approximation, no changes in internal energy are produced by the energy-flux convergence.

When the reference state is not steady, exact conservation of the domain-integrated energy may nevertheless be ensured by adding a constant to $\pi'$ that makes the domain integral of the first term in (4.3) exactly zero. This approach, which is discussed in the next section, does not guarantee strict energy conservation in every sub-domain, but the degree of non-conservation within individual sub-domains is likely to be small when both the domain-integrated energy is conserved and the first term in (4.3) is small. The smallest contribution to the other terms in (4.3) is the local rate of change of the kinetic energy density. Using our previous scaling arguments to compare the magnitude of the first term in (4.3) with the rate of change of kinetic energy, we again arrive at two cases. If the reference-state pressure gradients are no larger than those associated with $\pi'$, then

$$\frac{\dfrac{\pi'}{\tilde{\pi}}\dfrac{\partial}{\partial t}(\tilde{\rho}c_v\tilde{T})}{\dfrac{\partial}{\partial t}\left(\rho^*\dfrac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right)}=O\left(\frac{\sigma}{\min(1,\mathscr{R})}\right). \tag{4.4}$$

Suppose the reference-state pressure varies on a scale of $\tilde{L}=10^6$ m. For atmospheric circulations on scales where $L=10^4$ m and $\mathscr{R}>1$, the ratio in (4.4) is $10^{-2}$ and the first term in (4.3) might reasonably be neglected. On the other hand, for synoptic-scale motions with $L=10^6$ m and $\mathscr{R}=0.1$, the ratio in (4.4) increases to ten; however in this case the scaling estimate for $\pi'$ could be reduced from that given by (3.11) to that in (3.15) if the evolving reference-state pressure matched the full $\pi$ field except for small corrections arising for non-hydrostatic effects. If $L=10^6$ m and the pseudo-incompressible pressure is limited to that required to account for non-hydrostatic effects, the left-hand side of (4.4) is $O(10^{-5})$, and the first term in (4.3) could be confidently neglected.

## 5. Auxiliary relations

Other fundamental conservation laws can be easily derived for the pseudo-incompressible system using (1.5) in combination with the unapproximated momentum and thermodynamic equations. For example, the conservation of Ertel potential vorticity in isentropic flow takes the form

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}\cdot\nabla\theta}{\rho^*}\right) = 0, \tag{5.1}$$

where $\boldsymbol{\omega}$ is the vorticity with respect to an inertial coordinate system. This relation can be derived in a manner closely analogous to that for the full compressible system (Pedlosky 1987, p. 38). Since the momentum equations are unapproximated, Kelvin's circulation theorem applies without modification. The thermodynamic equation is also unapproximated, so the only change required from the derivation in Pedlosky is to note that the mass conserved by a material element of volume $V$ becomes $\rho^* V$ instead of $\rho V$.

The diagnostic pressure equation for the pseudo-incompressible system (3.4)–(3.6) is determined by multiplying the momentum equation by $\tilde{\rho}\tilde{\theta}$, taking the divergence of the result and subtracting $\partial/\partial t$ of (3.3) to obtain

$$\begin{aligned} c_p\nabla\cdot(\tilde{\rho}\tilde{\theta}\theta\nabla\pi') = {} & -\nabla\cdot(\tilde{\rho}\tilde{\theta}\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{k}\times\nabla_h(\tilde{\rho}\tilde{\theta}\boldsymbol{u}_h) \\ & + g\frac{\partial\tilde{\rho}\theta'}{\partial z} - c_p\nabla_h\cdot(\tilde{\rho}\tilde{\theta}\theta\nabla_h\tilde{\pi}_h) - \frac{\partial}{\partial t}\left(\frac{\tilde{\rho}H_m}{c_p\tilde{\pi}}\right) + \frac{\partial^2\tilde{\rho}\tilde{\theta}}{\partial t^2}. \end{aligned} \tag{5.2}$$

Note that the last term is an external forcing determined entirely by the evolution of the reference state pressure $\tilde{\pi}$. Numerical solutions to the pseudo-incompressible system would not typically involve the solution of the preceding equation for $\pi'$ at any given time level, but would rather require the solution of a similar elliptic equation to evaluate a pressure field that projects the evolving velocity field on to the subspace of velocity distributions satisfying (1.7) (Chorin 1968; Témam 1969; Durran 1999).

The solution for $\pi'$ obtained from (5.2) or the corresponding relation for the projection method only determines $\pi'$ to within a constant. That constant can be chosen to enhance the energy conservation properties of solutions with non-steady hydrostatic reference states by forcing the domain integral of first term in (4.3) to be exactly zero. Let $\pi''$ denote the solution arising from (5.2) and set $\pi' = \pi'' + C$, where $C$ is a constant. Let angle brackets indicate the integral over the domain, and define $\phi = (1/\tilde{\pi})\partial(\tilde{\rho}c_v\tilde{T})/\partial t$. Then choosing $C = -\langle\pi''\phi\rangle/\langle\phi\rangle$, will make $\langle\pi'\phi\rangle$ zero.

Once $\tilde{\pi}(\boldsymbol{x}, t)$ is specified, (3.4)–(3.6) and (5.2) form a system of five scalar equations and five unknowns that is closed without further use of the equation of state for perfect gases. Nevertheless, the equation of state would be needed to diagnose other thermodynamic quantities such as the sensible temperature, which would be required if moist processes were included in the governing equations. Consistent with the derivation of the energy equation (4.3), the equation of state for this system is the unapproximated perfect gas law, and therefore the temperature could be evaluated as $T = (\tilde{\pi} + \pi')\theta$.

## 6. Comparison with other acoustic-wave-filtered systems

Before discussing other anelastic systems used in the atmospheric sciences, consider the low-Mach-number approximation that has been employed in the simulation of mixtures of reactive gases (Majda 1984; McMurty *et al.* 1986). In this approach all the unknown variables are expanded in powers of $\mathscr{M}$; gravitational accelerations are neglected and the lowest-order balance in the momentum equation stipulates that $\nabla p^{(0)} = 0$, implying that the zeroth-order pressure field is a function only of time. For simplicity of comparison with the pseudo-incompressible result, suppose that the thermal diffusivity is zero, and let $H$ be the heating per unit volume. Then the lowest-order mass continuity and thermodynamic equations maybe be combined to yield

$$\nabla \cdot \boldsymbol{u} = \frac{1}{c_p p^{(0)}} \left( RH - c_v \frac{\mathrm{d}p^{(0)}}{\mathrm{d}t} \right) \tag{6.1}$$

which is the dimensional form of equation 3 in McMurty *et al.* (1986).

Now compare this to the pseudo-incompressible equation (3.3). When gravity is neglected, one may chose a reference state in which $\tilde{p}$ is spatially uniform, in which case $\tilde{\pi}$ and $\tilde{\rho}\tilde{\theta}$ will also be spatially uniform. Using the definition of $\pi$ and (3.7) the pseudo-incompressible equation for a spatially uniform basic state reduces to

$$\frac{\partial \tilde{\rho}\tilde{\theta}}{\partial t} + \tilde{\rho}\tilde{\theta} \nabla \cdot \boldsymbol{u} = \frac{H}{c_p \tilde{\pi}},$$

or

$$\frac{c_v}{c_p} \frac{\partial}{\partial t} (\ln \tilde{p}) + \nabla \cdot \boldsymbol{u} = \frac{H}{c_p \tilde{\rho} \tilde{T}},$$

which is equivalent to (6.1). Although the pseudo-incompressible and low-Mach-number approximations yield the same expression for the divergence of the velocity field, their full solutions are not equivalent. Indeed, a major difference between the two systems is the equation that is replaced by the condition on the velocity divergence. In the low-Mach-number approximation, (6.1) replaces the thermodynamic equation and a time-dependent mass continuity equation is retained, whereas the mass continuity equation is replaced by (3.3) and the thermodynamic equation is retained in the pseudo-incompressible system.

Other acoustic-wave-filtered equation sets have been derived by systematically expanding all unknown variables in terms of one or more small parameters (Shirgaonkar & Lele 2006; Bois 2006), and for one such expansion Klein (2000) has recovered the pseudo-incompressible system of Durran (1989) in the limit $\mathscr{M} \rightarrow 0$. The scale analysis presented in § 3 requires the Mach number to be small, but does not explicitly neglect the $O(\mathscr{M})$ contributions to most of the unknown variables. This of course does not imply that the pseudo-incompressible equations are more accurate than equation sets that systematically neglect all $O(\mathscr{M})$ perturbations, but retaining some of these small perturbations allows the pseudo-incompressible approximation to be exactly equivalent to the conceptually simple approximation to mass conservation embodied by (1.5).

Turning to previous approximations in atmospheric science, Lipps & Hemler (1982) (hereafter referred to as LH) and Bannon (1996*b*) have also derived energy-conservative anelastic systems without focusing as directly on the minimal criteria required to filter sound waves. For the fields considered in this paper: velocity, pressure and potential temperature, the prognostic equations in Bannon (1996*b*) are isomorphic to those in LH, and in the following, they will both be referred to as the LH system. The reference state about which buoyancy perturbations are computed in the LH system varies only in the vertical. When the reference state is only a function of $z$, (3.3)–(3.6) reduce to the system in Durran (1989) (hereafter referred to as D).

Several comparisons between the LH and D systems have appeared in the literature. Nance & Durran (1994) examined linear mountain-wave solutions in an isothermal atmosphere and noted that the LH system provides a perfect prediction of the vertical wavelength and the vertical velocity field in hydrostatic waves (the D system does not), but the D system provides a better approximation to the horizontal velocity field. In addition the errors in the LH system increase and those in the D system decrease as the perturbations become more non-hydrostatic. Consistent with the superiority of the D system in non-hydrostatic applications, Nance (1997) computed resonant wavelengths in trapped mountain lee waves using the full compressible equations and the LH and D systems and found the D system gives more accurate solutions.

Bannon (1996*a*) computed solutions to Lamb's adjustment problem and found the LH system superior to D. Again this is consistent with the findings in Nance & Durran (1994) that LH is superior to D for computing vertical velocities in hydrostatic problems, since the thermal forcing in Lamb's adjustment problem is uniform in the horizontal. Almgren (2000) showed that exact solutions to Lamb's adjustment problem can be obtained using the D system with a time-evolving reference state, observing that such spatially uniform heating is more properly absorbed by the reference state, rather than the perturbational fields. The generalization of the D system presented in § 3 allows for such changes in the reference state.

Nance & Durran (1994) also evaluated numerical solutions for linear and nonlinear flow over topography or by an elevated heat source using the compressible equations and several anelastic systems. They concluded that both the D and LH systems were superior to other anelastic systems that have been used in atmospheric models, but that in those cases where analytic solutions were available for comparison, the difference between the D and LH solutions was less than the errors introduced by discretizing the equations and applying approximate open boundary conditions in the numerical models.

Davies *et al.* (2003) performed a normal mode analysis comparing the fully compressible equations with various approximate acoustic-wave-filtered systems, including LH, D and the hydrostatic compressible system. They examined the influence of each approximation on the frequency of the mode, height-scale distortion, the partitioning of energy among the individual perturbation fields, and the vertical levels at which nodal lines appear. Their results show that the pseudo-incompressible system performs better than the other systems, except in the case of very deep hydrostatic modes, for which the hydrostatic compressible system gives the best results. Yet, not surprisingly, the compressible hydrostatic system does not adequately represent non-hydrostatic motions. It is possible that the difficulties with deep hydrostatic modes can be avoided if, as suggested in § 3, the pseudo-incompressible pressure is computed as a perturbation to a time-and-space-varying hydrostatic reference state. Most recently, Smolarkiewicz & Dörnbrack (2008) discussed the differences between consistent numerical solutions to the LH and D systems across a wide range of scales and found significant differences in deep domains, but did not determine which solution was closer to the correct compressible result.

## 7. Conclusions

Classical sound waves propagate through isentropic pressure perturbations. Following Durran (1989), it is proposed that mass conservation in stratified compressible fluids at low Mach number be enforced with respect to a pseudo-density $\rho^*$ that is not influenced by isentropic changes in the pressure. A hypothetical fluid, in which the actual density behaves like such a pseudo-density, will act as if it were incompressible when the flow is isentropic, and will not support sound waves.

Here we have shown that $\rho^*(S, x, y, z, t)$ is the most general functional form for the pseudo-density that can be used in the mass conservation relation (1.5) to eliminate sound waves in single-constituent fluids. The same result holds for multiple-constituent fluids of fixed composition, such as dry air, in which any thermodynamic state variable may be expressed in terms of two other state variables.

For perfect gases, the choice $\rho^* = \tilde{\rho}\tilde{\theta}/\theta$ allows the construction of an energy conservative system (3.3)–(3.6) in which the momentum and thermodynamic equations are unapproximated and (1.5) reduces to a diagnostic equation for the scaled velocity divergence. The total energy equation for this system differs from that in the full compressible system in that the true density is replaced by $\rho^*$ in the expressions for the energy-flux and mechanical energy densities, and the total energy-flux convergence produces changes in mechanical but not internal energy. Any changes in internal energy are due to the time evolution of the reference-state pressure.

The hydrostatic reference state for the pseudo-incompressible system has been generalized to include low-frequency long-wavelength variations in time and space. Scale analysis gives the criterion $\mathscr{M}^2 \ll \min(1, \mathscr{R}^2)$ for the validity of the pseudo-incompressible system (3.3)–(3.6) in circumstances where the reference-state pressure gradients are no larger than those associated with the pseudo-incompressible pressure $\pi'$. This criterion is easily satisfied by small-scale atmospheric motions that are not significantly influenced by the Earth's rotation, and is also just satisfied by quasi-geostrophic synoptic-scale motions. The pseudo-incompressible system could be extended to planetary-scale motions if the hydrostatic reference state were concurrently integrated using a compressible hydrostatic model with sufficient accuracy that the reference-state pressure perturbations on scales larger than $O(1000)$ km dominated the pseudo-incompressible pressure perturbations at those same scales. In such cases the pseudo-incompressible solution would not add important information about the planetary-scale flow, but the resulting model architecture would have the potential to efficiently simulate all scales of low-Mach-number atmospheric motion. The numerical details required to coordinate the reference-state and pseudo-incompressible solutions are left to future research, along with the incorporation of moist processes (e.g. Klein & Majda 2006).

Gravity is the only body force considered in the preceding, but these results can be easily generalized to include other conservative body forces. Moreover, because the pseudo-incompressible system avoids all approximations in the momentum equations and includes only a minor approximation to the heating term in the thermodynamic equation, it is likely that similar formulations can be constructed for a variety of different applications involving low-Mach-number disturbances in perfect gases. Such formulations might be regarded as generalizations of the Boussinesq approximation; however since they avoid making approximations in the momentum equations, they also constitute generalizations of the basic concept of incompressibility to low-Mach-number compressible stratified flow.

The author is grateful for helpful discussions with Chris Bretherton, Piotr Smolarkiewicz, Akio Arakawa and Greg Hakim and for insightful comments from three anonymous reviewers. This research was supported by National Science Foundation grant ATM-0506589.

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